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Frames in Compressive Sensing and Approximate Signal Recovery Pertaining to Physical Sensing Matrices

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14. ABSTRACT
In a series of papers we made a deep study of phase retrieval which has a broad spectrum of engineering applications and even will be needed to align the mirrors of the new James Webb Space Telescope. In some applications, such as crystal twinning, we have to recover the phase of a signal from its projections onto subspaces of the space. It was believed that projections give much less information than vectors and so it should take many more projections than vectors to do phase retrieval. We proved the surprising result that actually, it takes fewer projections than vectors to do phase retrieval. We introduced two new areas of research: Integer frames and weaving frames, and developed their basic properties. Weaving frames have application for pre-processing signals. Integer frames have the major advantage that they can process signals without the first level of quantization errors since they do not need approximations to their coefficients.

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(1) **Phase retrieval by projections:** In [10] we see the solution to a 50 year old engineering problem with applications to x-ray crystallography, x-ray tomography, crystal twinning, and much more. The results here on phase retrieval by projections are truly significant. Previous work indicated that it would take nlog(n) arbitrary rank projections to do phase retrieval. We showed that 2n-1 projections is sufficient in the real case and surprised everyone even more by giving evidence that actually fewer than 2n-1 projections might work - an unimaginable result with the prospect of major applications. We are continuing work in this direction to find the least number of projections needed and to do the complex case and are just starting to work directly with the engineers for implementation.
(2) In [11] we introduced yet another new area of research in frame theory: **Integer frames**. The idea here is to speed up calculations and accuracy in applications of frame theory. Integer frames are Hilbert space frames for which all their coefficients are integers. The advantages here include: (a) We do not need to *approximate* the frame coefficients speeding up calculations. (b) Quantization sometimes becomes a problem when there are multiple levels of processing and this eliminates one level of quantizations for the process. This is an ongoing project because many of the examples constructed in [11] are *sparse frames* and we really need non-sparse frames - and preferable something close to equiangular frames.
(3) In [12] we made the first systematic study of **outer product frames**. These frames have the potential for serious applications but are very difficult to work with. One major result shows that equiangular tight frames give the absolute best frame bounds for outer product frames. We expect this paper to open up this area to serious applications. This is also an ongoing project. There are a number of critical construction problems left here which need to be resolved in order to make this class of frames really usable.
(4) In [9], we made the first systematic study of the **distribution of frame coefficients** of Hilbert space frames. This is fundamental to this field and should have been done 20 years ago, but no one had successfully tackled this area. Our work is comprehensive and

invaluable. This project is also ongoing because we need to do *random methods* here where the outcomes will be significantly better than currently available.

(5) In [2] we surprised everyone by showing that **frame expansions** of Parseval frames are 1-unconditional series and then giving a complete classification of frames with this property. This project is complete.

(6) In [3] we introduce a new topic: *weaving frames*. This topic has the potential for important applications in pre-processing signals and dealing with multiple wireless sensor networks. This paper is a comprehensive study and should generate parallel research and applications. This is an ongoing research project since for applications we need to understand *weaving* for Gabor frames.

(7) [4] is a continuation of the paper by Bodmann Cahill and Casazza, Fusion frames and the restricted isometry property. (English) Numer. Funct. Anal. Optim. 33, No. 7-9, 770-790, 2012.

In analogy with this preceding paper, the correspondence between RIP matrices and fusion frames obtained by a partial orthonormalization strategy is at the core of the results. In contrast to the preceding paper, a specific construction of RIP matrices by random Gaussian matrices is used in order to derive stronger consequences for the resulting fusion frames consisting of independent, uniformly distributed subspaces. A measure concentration argument shows that the Hilbert-Schmidt inner products between the orthogonal projections onto the random subspaces concentrate near an average value. Overwhelming success probability for near tightness and equiangularity is guaranteed if the dimension of the subspaces is sufficiently small compared to that of the Hilbert space and if the dimension of the Hilbert space is small compared to the sum of all subspace dimensions.

(8) The paper [5] paper leverages the results on the nearly equal norm and nearly tight fusion frames to derive error bounds for packet encodings in the presence of data loss (erasures). The fusion frames encode a vector in a Hilbert space in terms of its components in subspaces, which can be identified with packets of linear coefficients. The fusion frame performance is evaluated under the assumption that the vector to be transmitted is uniformly distributed on the sphere and when part of the packets is transmitted perfectly and another part is lost in an adversarial, deterministic manner. The performance is measured by the mean-squared Euclidean norm of the reconstruction error when averaged over the transmission of all unit vectors. The main result is that a random selection of fusion frames performs nearly as well as previously known optimal bounds for the error, characterized by optimal packings of subspaces, which are known not to exist in all dimensions.

(9) [7] Shearlets are a type of frames that provide directional multiscale representations with optimal sparsity properties for a class of piecewise smooth signals with piecewise smooth singularities. The goal of this paper is to show that a variant of shearlets can be constructed with the help of standard wavelet filters and standard Gabor windows. Gabor shearlets, however, are based on a different group representation than previous shearlet constructions. Unlike the usual shearlets, the new construction can achieve a redundancy as close to one as desired. In combination with Meyer filters, the cone-adapted Gabor shearlets constitute a tight frame and provide low-redundancy sparse approximations.

(10) [16] gives constructions of tight and full spark Chebyshev frames from truncations of Vandermonde-like matrices of orthogonal polynomials are presented. These are frames with real entries. Worst case coherence analyses are also carried out. We show that for sufficiently high degree, the minimum angle between distinct frame vectors is bounded below by about 44 degrees. In a related study, we also provide a worst case coherence analysis for equal norm tight $M \times N$ frames from truncated DFT matrix. The cosine of the smallest angle between these distinct frame elements is asymptotic to $\frac{\sin(\pi\alpha)}{\pi\alpha}$ where $\alpha = M/N$.

(11) [17] In this article, a fast and super resolution method for direction of arrival (DOA) estimations is proposed under low signal-to-noise ratio using a limited number of snapshots (of measurements). The method is based on a sparse signal reconstruction technique of null space tuning in the context of compressed sensing and sparse representations. A crucial correlation operation is proposed to mitigate the noise effect from the sparse representation point of view. The proposed method has the characteristics of simultaneous high resolution, robustness to noise, and effective at estimating number of sources. The algorithm is also computationally efficient.

(12) [18] In array signal processing, signal detections and direction-of-arrival (DOA) estimations, for many years, suffer from poor resolutions when several signals/targets are close to each others. In these problems, observed signals are naturally sparse linear combinations of array direction vectors sampled at arriving angles. The matrix A of array direction vectors sampled at (typically) large number of angle points is a large redundant frame with columns being frame vectors. If the dimension of the ambient vector space is n, and the column of A is N. We have shown that there is a common sparsity-inducing dual frame for a set of n linearly independent vectors. Array direction matrix A is typically full spark. Consequently, sparsity-inducing duals can be evaluated a priori for any blocks of n columns of A. It becomes therefore particularly effective for DOA estimations when signals are clunked together. This approach does

not need the typical ℓ_1-min procedure, as applying sparse duals to observed signals yields directly the few non-zero coefficients (signals), under a multiresolution pursuing strategy.

(13) [19] This article is about a group of fast iterative and thresholding algorithms for sparse signal recovery. The algorithms are faster and more effective than most, if not all, known algorithms. Analytical convergence results are also established. One core algorithm is shown to converge in finite many steps. Quite convincing examples are shown through numerical experiments. The algorithms are particularly effective for large scale problems.

(14) [20] This article introduces and studies the notion of sparse dual frames for the first time. Sparse dual frames in this context refer to dual frames of smaller or smallest nonzero vectors and nonzero entries within a vector. Theoretical lower bound of the sparsity for random frames and Gabor dual frames are obtained. Gabor dual frames of the smallest time and frequency support are obtained. Analysis of sparse dual Gabor frames results in new duality results which are monumental for the sparse dual frame analysis in the case of dual Gabor frames.

(15) [21]

When signals have sparse (coherent) frame representations, i.e., signals are sparse with respect to coherent frames, the compressed sensing problem becomes much more complicated. We have seen that a sparse-dual-frame based ℓ_1-analysis approach for the sparse signal recovery is the most effective one among all known methods, such as the ℓ_1 synthesis approach and the conventional ℓ_1-analysis approach. Meantime, we also have an observation about the error bound of the sparse-dual-based approach. The alternating iterative algorithm also converges fine for all numerical tests we observed. The convergence guarantee is what we are trying to establish. We have made a fundamental observation that the convergence is fundamentally a numerical stability issue of a new "sparse-dual-infused" system matrix $\tilde{A}$ (as in equation (2) in a later description). The next step is to establish the conditions with which "tails" of the coefficients are diminishing.

(16) [22] Compressed sensing with frames is typically formulated to recover f from the under-determined measurements $y = Af$ (with or without noise), assuming $f = Dx$. Here x is sparse and D is a frame. One typical and relatively successful (and natural) approach is to simply write the measurement as $y = ADx$, and trying to recover the coefficient x by various means. There is, however, no good performance guarantee to such an approach, because of the coupling of A and D in their product. We have seen that the sparse-dual-based

analysis approach can be written as

$$\min_v \|v\|_1 \quad \text{subj. to} \quad \tilde{A}v = \tilde{b}, \tag{1}$$

where

$$\tilde{A} \equiv \begin{bmatrix} AD \\ I - D^*(DD^*)^{-1}D \end{bmatrix}, \quad \text{and} \quad \tilde{b} \equiv \begin{bmatrix} b \\ \Delta x \end{bmatrix}. \tag{2}$$

Here Δx is the tail difference between the canonical frame expansion coefficients and the sparse expansion coefficients.

We have observed that such a sparse-dual-based analysis approach, in the form of (1) and (2), has a "decoupling" functionality to decouple the product AD in all traditional performance guarantee. For instance, that the unique ℓ_1 solution is guaranteed the ℓ_0 solution as well if and only if $\tilde{A}$ has the *Null Space Property* (NSP). NSP is entirely about the null space of $\tilde{A}$. In the original problem of solving x from $y = ADx$, the NSP would be about the null space of AD. A close examination shows that

$$ker(\tilde{A}) = \{v \mid Dv \neq 0, Dv \in ker(A)\} = D^*(DD^*)^{-1}(kerA).$$

As a result, the kernel of $\tilde{A}$ is no longer about the kernel of AD, but a mapping of $kerA$ only. Hence, the sparse-dual-analysis approach decouples A from the product with D in the traditional approach of solving for x from $y = ADx$.

We believe that this observation is capable to provide much more satisfactory theoretical insight about the sparse recovery problems when signals are sparse with respect to frames.

References

[1] S.B. Bahmenpour, J. Cahill, P.G. Casazza, J. Jasper, and L. Woodland, *Phase retrieval signal reconstruction by projections*, Preprint.

[2] T. Bemrose, P.G. Casazza, and R. Lynch, *The unconditional constant for Hilbert space frame expansions*, Submitted.

[3] T. Bemrose, P.G. Casazza, and R. Lynch, *Weaving Frames*, Preprint.

[4] B.G. Bodmann, *Random fusion frames are nearly equiangular and tight*, Linear Algebra and its Applications, **439**, Issue 5, (2013), Pages 1401-1414, ISSN 0024-3795, http://dx.doi.org/10.1016/j.laa.2013.04.024.

[5] B.G. Bodmann and K. Singh,

[6] Random fusion frames for loss-insensitive packet encoding, Proc. SPIE 8858, Wavelets and Sparsity XV, 88580F/1-12, (2013).

[7] B.G. Bodmann, G. Kutyniok, X. Ahuang,

[8] Gabor shearlets, Appl. Comput. Harmon. Anan. To appear.

[9] K. Brewster, P.G. Casazza, and L. Woodland, *The distribution of Hilbert space frame coefficients*, arXiv:

[10] J. Cahill, P.G. Casazza, J. Peterson and L. Woodland, *Phase retrieval by projections*, To appear Houston J. of Math.

[11] P.G. Casazza, R. Lynch, J. Tremain, and L. Woodland, *Integer Frames*, To appear: Houston J. Math.

[12] P.G. Casazza, E. Pinkham, and B. Thompson, *Outer Product Frames*, Preprint.

[13] P.G. Casazza and J.C. Tremain, *Consequences of the Marcus/Spielman/Srivastava solution to the Kadison-Singer Problem*, arXiv: 1407:4768.

[14] P.G. Casazza and L. Woodland, *Phase retrieval by vectors and projections*, To appear: Contemp. Math.

[15] P.G. Casazza and L. Woodland, *The fundamentals of spectral tetris frame constructions*, Submitted.

[16] S. Li, D. Ellis and E. Hauashi, *Tight and Full Spark Chebyshev Frames with Real Entries and Worst-Case Coherence Analysis* Submitted for publication, June, 2014.

[17] S. Li, C. Zeng, *A fast and super resolution method in DOA estimation under severe noise* Revised and submitted for publication, May, 2014.

[18] S. Li and L. Wang, *Sparsity inducing dual frames applied in MIMO radar signal detections* Article in preparation.

[19] S. Li, T. Mi, and Y. Liu, *Fast thresholding algorithms with feedbacks for sparse signal recovery*, *Applied and Computational Harmonic Analysis*, **37**, No.1, pp 69-88, (2014).

[20] S. Li, T. Mi and Y. Liu, *Sparse Dual Frames and Dual Gabor Functions of Minimal Time and Frequency Supports,* Journal of Fourier Analysis and Applications, **19**(1), pp 48-76, (2013).

[21] J. Cahill, P.G. Casazza S. Li, T. Mi, *Convergence of an alternating iterative algorithm for the sparse-dual-based ℓ_1-analysis method.* This project is in progress.

[22] J. Cahill, P.G. Casazza and S. Li, *The decoupling effect of sparse-dual-based ℓ_1 analysis approach in sparse signal recovery with frames.* This project is in progress.
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